FILTERING RESPONSES OF SELECTED DISTANCE-DEPENDENT WEIGHT FUNCTIONS[1]

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ABSTRACT

The filtering response of the Cressman weight functions and a truncated low-pass filter are presented as functions of influence radius.

1. INTRODUCTION

An important interim step in the preparation of a numerical forecast is the assignment of estimates of atmospheric variables to points of a regular net made from observations taken at widely-spaced, irregularly-distributed reporting stations. A multiple-scan interpolation method with varying influence radii has been outlined by Cressman [1]. The present note examines the filtering response of the Cressman weight functions and of a truncated low-pass filter. The interpretation is based upon the similarity between the operations of smoothing and filtering.

2. TWO-DIMENSIONAL FILTER RESPONSES

Consider the linear smoothing operation

$$\bar{f}(x, y)=\iint_{-\infty}^{\infty} \omega(x', y')f(x+x', y+y')dx'dy' \tag{1}$$

where $f(x, y)$ is the total (signal plus noise) field and $\omega(x', y')$ is the impulsive response, or weight function, normalized for unit response at wave number zero. Also, $\omega(x', y')$ must be square-integrable over all (x', y') to insure that the operation is one of smoothing. In application, equation (1) must be approximated by numerical cubature. It is assumed that the data are sufficiently dense so that the continuum solution is well estimated by the discrete analog.

When $f(x, y)$ may be represented by (1), then its spectral density of variance (or power) $P(m, n)$ is related to the corresponding filtered quantity by

$$\bar{P}(m, n)=4\pi^2|Y(m, n)|^2P(m, n). \tag{2}$$

The quantities m and n represent wave numbers $2\pi/L_x$ and $2\pi/L_y$, respectively. $Y(m, n)$ is the Fourier transform of $\omega(x', y')$ and is termed the transfer function.

Sasaki [3] has used a square wave-number cell in developing a low-pass filter. Petersen and Middleton [2] have concluded that the best basic cell shape in wave-number domain for the two-dimensional low-pass filter is hexagonal. The cell shape chosen for further examination here is a circular region.

With the presumption of non-overlapping signal and noise spectra, the weight function which filters all noise of wave number $k(=\sqrt{m^2+n^2})$ greater than some k_0 has a transfer function given by

$$Y(k)=\begin{cases} \dfrac{1}{2\pi}, & 0\leq k<k_0 \\ \dfrac{1}{4\pi}, & k=k_0 \\ 0, & k>k_0. \end{cases} \tag{3}$$

The weight function corresponding to (3) is a function of $r(=\sqrt{x'^2+y'^2})$ only. As shown by Sneddon [4], when $Y=Y(k)$ only, then $Y(k)$ and $\omega(r)$ are inverse Hankel transforms of order zero:

$$\begin{aligned} Y(k)&=\int_0^\infty \omega(r)rJ_0(kr)dr \\ \omega(r)&=\int_0^\infty Y(k)kJ_0(kr)dk. \end{aligned} \tag{4}$$

Here $J_0(kr)$ is a Bessel function of the first kind and order zero.

The weight function corresponding to (3) is

$$\omega(r)=\frac{k_0}{2\pi}\frac{J_1(k_0r)}{r}, \tag{5}$$

where $J_1(k_0r)$ is a Bessel function of the first kind and order one.

As pointed out by Sasaki [3], the nature of the atmospheric data distribution makes the use of oscillatory weight functions somewhat hazardous. Certainly, the interval of integration indicated must be finite. In this instance, a weight function ω_T might be defined as

$$\omega_T(r)=\begin{cases} \dfrac{k_0}{2\pi}\{1-J_0(k_0R)\}^{-1}\dfrac{J_1(k_0r)}{r}, & 0\leq r\leq R \\ 0, & r>R \end{cases} \tag{6}$$

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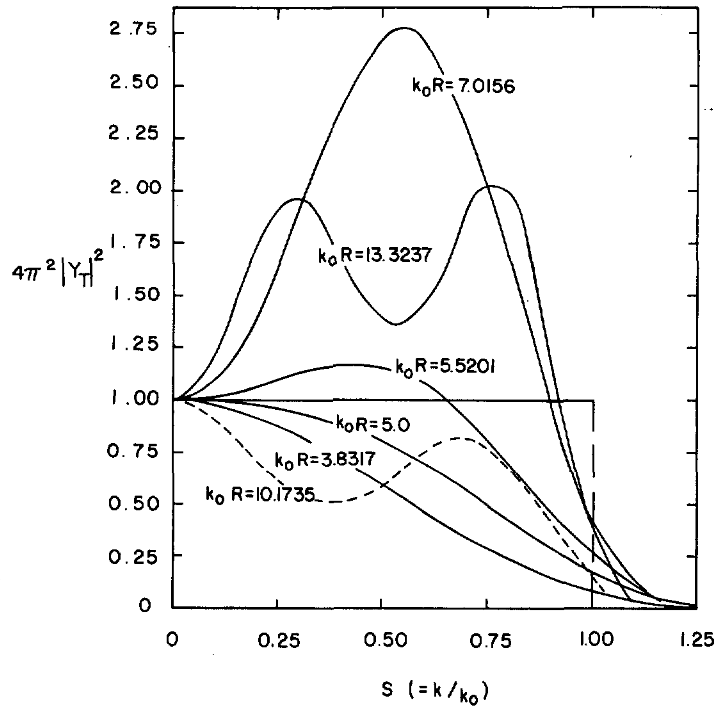

FIGURE 1.—Power transfer function for the truncated low-pass filter as a function of (k/k_0).

where R is the influence radius. The corresponding transfer function is

$$Y_T(sk_0)=\frac{1}{2\pi[1-J_0(k_0R)]}\int_0^{k_0R} J_1(y)J_0(sy)dy \tag{7}$$

where $s=k/k_0$.

Equation (7) has been approximated with the trapezoidal rule using $\Delta y \leqq 0.05$ and $s=0.1[0.1]5$. The power transfer function, $4\pi^2|Y_T(k)|^2$, is shown in figure 1 for various choices of k_0R.

Although the cut-off wave number k_0 is only nominal when the influence radius is finite, the secondary lobes of the power transfer function at high wave numbers have amplitudes of less than 10^{-2} and are rapidly damped with increasing s. The choice of $k_0R=3.8317$ (the first zero of $\omega_T(k_0, r)$) shows that the major half-lobe of ω_T is sufficient to reconstitute the longest wavelengths. A choice of the second (7.0156) or fourth (13.3237) null points of ω_T gives unacceptable amplification of the shorter wavelengths. The third null point (10.1735) would yield reasonable short-wave representation, but at the expense of longer wavelengths. Two intermediate values of k_0R (5.0 and 5.5201) are shown to illustrate continuity of the results. It would appear that a value of k_0R between 5.0 and 5.5 would yield the best filter. However, because of the discrete, random nature of the actual data distribution, the inclusion of even a small portion of a negative lobe of ω_T may have effect of marked amplification of short waves.

The normalized Cressman [1] weight function is

$$\omega_c(R,r)=\begin{cases}\left\{\dfrac{1}{\pi R^2(\ln 4-1)}\right\}\left\{\dfrac{R^2-r^2}{R^2+r^2}\right\}, & 0\leqq r\leqq R\\ 0, & r>R.\end{cases} \tag{8}$$

Now let a real, positive parameter k_0 be defined by $k_0R=3.8317$. Equation (8) becomes

$$\omega_c(k_0,r)=\begin{cases}\left\{\dfrac{k_0^2}{14.682(\ln 4-1)}\right\}\left\{\dfrac{14.682-(k_0r)^2}{14.682+(k_0r)^2}\right\}, & 0\leqq k_0r\leqq 3.8317\\ 0, & k_0r>3.8317.\end{cases} \tag{9}$$

It might be noted that ω_c, determined experimentally, is nearly coincident with ω_T truncated at $k_0r=3.8317$. Their filtering properties are necessarily similar.

When the weight function is constant, the power transfer function is

$$4\pi^2|Y(k)|^2=4\left[\frac{J_1(kR)}{kR}\right]^2. \tag{10}$$

Waves whose lengths are comparable to $2R$ or shorter are virtually eliminated. The power in waves of length $10R$ or greater is reproduced to better than 90 percent.

3. CONCLUSIONS

The notion of analyzing in scales, sometimes used qualitatively with the Cressman weight and varying influence radius with successive scans, may be made explicit. The specification of R determines a nominal k_0 and the spectral density modification. The total effect of repeated scans with variable influence radii may also be inferred for R large enough to insure a large data sample since the associated power transfer functions are multiplicative. Although the constant weight function sharply filters short waves for an information continuum, limited data samples for small R may have the opposite response. Lack of information corresponds to zero weighting, and with arbitrary data spacing almost any response may be generated.

REFERENCES

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